Kuwait University
Department of Mathematics and Computer Science

Math 102 : Calculus II	**Summer Semester 2009**	**Final Examination**

Saturday 15 August 2009

Duration: 120 minutes — Total marks: 40

Justify all your answers

1. Let $f(x) = 3x^3 - 2x^{-1} - 1$, $x > 0$.
   (a) Show that f is one-to-one on $(0, \infty)$. [1 mark]
   (b) Find the domain and range of f^{-1}. [1 mark]
   (c) Find $(f^{-1})'(0)$. [2 marks]

2. Consider $f(x) = (1-x)^{\ln(2x+1)}$.
   (a) Find the domain of f. [1 mark]
   (b) Find $f'(1/2)$. [2 marks]

3. Find the value of $a > 0$ for which $\lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^x = 9$. [4 marks]

4. Evaluate the following integrals.
   (a) $\int \ln\left(1+\sqrt{x}\right)\,dx$. [3 marks]
   (b) $\int \frac{\cos^3 x + 2\cot x}{\sin^2 x}\,dx$. [3 marks]
   (c) $\int (x^2+4x+3)^{-3/2}\,dx$. [3 marks]

5. Test $\int_1^\infty \frac{1}{x(x^2+4)}\,dx$ for convergence and evaluate it if it is convergent. [5 marks]

6. Consider the curve C parametrized by $x = \frac{1}{2}\ln(1-t^2)$ and $y = \arccos t$ for $0 \le t \le 3/4$.
   (a) Find the length of C. [3 marks]
   (b) Find an equation of the tangent line at the point corresponding to $t = 1/\sqrt{2}$. [2 marks]

7. Find the centroid of the region bounded by the curves $y = \sec^2 x$, $y = 0$, $x = 0$ and $x = \pi/4$. [5 marks]

8. Find the area inside the polar curve $r = 1 - \cos\theta$ and outside the polar curve $r = \cos\theta$. [5 marks]

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ANSWERS

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1. (a) $f'(x) = 9x^2 + 2x^{-2} > 0$ for $x > 0$. So f is increasing, and consequently one-to-one on $(0, \infty)$.

   (b) $f(x) \to -\infty$ as $x \to 0^+$, and $f(x) \to \infty$ as $x \to \infty$. Since f is continuous on its domain $(0, \infty)$, this implies that its range is $(-\infty, \infty)$.
   The domain of f is the range of f^{-1}, and vice versa.
   Thus the domain of f^{-1} is $(-\infty, \infty)$ and the range of f^{-1} is $(0, \infty)$.

   (c) When $x = 1$, $f(x) = 0$. So $f^{-1}(0) = 1$, and
   $$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(1)} = \frac{1}{9(1)^2 + 2(1)^{-2}} = \frac{1}{11}.$$

2. (a) The exponent is defined if and only if $2x + 1 > 0$, i.e. $x > -1/2$. Additionally one needs $1-x > 0$, $1-x = 0$ and $\ln(2x+1) > 0$, or, $1-x \neq 0$ and $\ln(2x+1) = r$ where r is a rational number with odd denominator, i.e. $x < 1$, $x = 1$, or $x = (e^r - 1)/2$ with r as stated.
   Answer: $(-1/2, 1] \cup \{(e^r - 1)/2 : r$ is a rational number with odd denominator$\}$.

   (b) Use logarithmic differentiation.
   $$\ln[f(x)] = \ln(2x+1)\ln(1-x)$$
   $\Longrightarrow$
   $$\frac{f'(x)}{f(x)} = \frac{2}{2x+1}\ln(1-x) - \frac{1}{1-x}\ln(2x+1)$$
   $\Longrightarrow$
   $$\frac{f'(1/2)}{f(1/2)} = \ln(1/2) - 2\ln 2 = -3\ln 2$$
   $\Longrightarrow$
   $$f'(1/2) = -3(\ln 2)f(1/2) = -3(\ln 2)(1/2)^{\ln 2} = -3(\ln 2)2^{-\ln 2}.$$

3. The question is equivalent to finding the value of $a > 0$ for which
   $$\lim_{x\to\infty} x \ln\left(\frac{ax+1}{ax-1}\right) = \ln 9 = 2\ln 3$$
   or
   $$\begin{aligned} 2\ln 3 &= \lim_{x\to\infty} \frac{\ln\left(\frac{ax+1}{ax-1}\right)}{x^{-1}} = \lim_{x\to\infty} \frac{\ln\left(\frac{a+x^{-1}}{a-x^{-1}}\right)}{x^{-1}} = \lim_{z\to 0^+} \frac{\ln\left(\frac{a+z}{a-z}\right)}{z} \\ &= \lim_{z\to 0^+} \frac{\ln(a+z) - \ln(a-z)}{z}. \end{aligned}$$
   Since the last limit is of the indeterminate type $0/0$, the question can be further reduced by L'Hospital's Rule to finding $a > 0$ for which
   $$2\ln 3 = \lim_{z\to 0^+} \frac{\frac{d}{dz}[\ln(a+z) - \ln(a-z)]}{\frac{d}{dz}z} = \lim_{z\to 0^+} \frac{\frac{1}{a+z} + \frac{1}{a-z}}{1} = \frac{2}{a}.$$
   Answer: $a = 1/\ln 3$.

4. (a) Substitute $t = 1 + \sqrt{x}$. So $x = (t-1)^2$ and $dx = 2(t-1)\,dt$. This gives

$$\int \ln\left(1+\sqrt{x}\right)\,dx = \int 2(t-1)\ln t\,dt.$$

Integrate by parts with $u = \ln t$ and $dv = 2(t-1)\,dt$. So $du = (1/t)\,dt$ and $v = t^2 - 2t$. This gives

$$\begin{aligned}
&\int \ln\left(1+\sqrt{x}\right)\,dx \\
&= (t^2-2t)\ln t - \int \frac{t^2-2t}{t}\,dt = t(t-2)\ln t - \int (t-2)\,dt \\
&= t(t-2)\ln t - \tfrac{1}{2}t^2 + 2t + C \\
&= \left(1+\sqrt{x}\right)\left(\sqrt{x}-1\right)\ln\left(1+\sqrt{x}\right) - \tfrac{1}{2}\left(1+\sqrt{x}\right)^2 + 2\left(1+\sqrt{x}\right) + C.
\end{aligned}$$

Redefining the constant of integration, the answer simplifies to

$$\int \ln\left(1+\sqrt{x}\right)\,dx = (x-1)\ln\left(1+\sqrt{x}\right) - \tfrac{1}{2}x + \sqrt{x} + C.$$

(b)

$$\begin{aligned}
\int \frac{\cos^3 x + 2\cot x}{\sin^2 x}\,dx &= \int \left(\frac{\cos^3 x}{\sin^2 x} + \frac{2\cos x}{\sin^3 x}\right)dx \\
&= \int \left(\frac{1-\sin^2 x}{\sin^2 x} + \frac{2}{\sin^3 x}\right)\cos x\,dx \\
&= \int \left(\frac{1}{\sin^2 x} - 1 + \frac{2}{\sin^3 x}\right)\cos x\,dx \\
&= -\frac{1}{\sin x} - \sin x - \frac{1}{\sin^2 x} + C \\
&= -\sin x - \csc x - \csc^2 x + C.
\end{aligned}$$

(c) Completing the square, $x^2 + 4x + 3 = (x+2)^2 - 1$. This suggests substituting $x + 2 = \sec\theta$. So $(x^2+4x+3)^{1/2} = \tan\theta$ and $dx = \tan\theta\sec\theta\,d\theta$. This gives

$$\begin{aligned}
\int (x^2+4x+3)^{-3/2}\,dx &= \int \frac{\tan\theta\sec\theta}{\tan^3\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = -\frac{1}{\sin x} + C \\
&= -\frac{\sec\theta}{\tan\theta} + C = -(x+2)(x^2+4x+3)^{-1/2} + C.
\end{aligned}$$

5. Consider

$$\begin{aligned}
\int_1^t \frac{1}{x(x^2+4)}\,dx &= \frac{1}{4}\int_1^t \left(\frac{1}{x} - \frac{x}{x^2+4}\right)dx = \frac{1}{4}\left[\ln x - \frac{1}{2}\ln(x^2+4)\right]\Bigg|_1^t \\
&= \frac{1}{8}\left[2\ln t - \ln(t^2+4) - 2\ln 1 + \ln 5\right] = \frac{1}{8}\ln\left(\frac{5t^2}{t^2+4}\right) \\
&= \frac{1}{8}\ln\left(\frac{5}{1+4t^{-2}}\right) \to \frac{1}{8}\ln\left(\frac{5}{1}\right) \qquad \text{as } t \to \infty.
\end{aligned}$$

Hence the integral is convergent, and its value is $(\ln 5)/8$.

6. (a)
$$\frac{dx}{dt} = -\frac{t}{1-t^2} \quad \text{and} \quad \frac{dy}{dt} = -\frac{1}{\sqrt{1-t^2}}$$

$\Longrightarrow$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{t^2}{(1-t^2)^2} + \frac{1}{1-t^2} = \frac{1}{(1-t^2)^2}.$$

So the length is

$$\begin{aligned}\int_0^{3/4} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_0^{3/4} \frac{1}{1-t^2}\, dt \\ &= \frac{1}{2}\int_0^{3/4} \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt \\ &= \tfrac{1}{2}\left[\ln(1+t) - \ln(1-t)\right]\Big|_0^{3/4} \\ &= \tfrac{1}{2}\left[\ln(7/4) - \ln(1/4)\right] = \tfrac{1}{2}\ln 7.\end{aligned}$$

(b) The point (x_1, y_1) on the curve corresponding to $t = 1/\sqrt{2}$ is given by

$$x_1 = \tfrac{1}{2}\ln(1/2) = -\tfrac{1}{2}\ln 2 \quad \text{and} \quad y_1 = \arccos(1/\sqrt{2}) = \pi/4.$$

The slope of the tangent line at any point is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sqrt{1-t^2}}{t}.$$

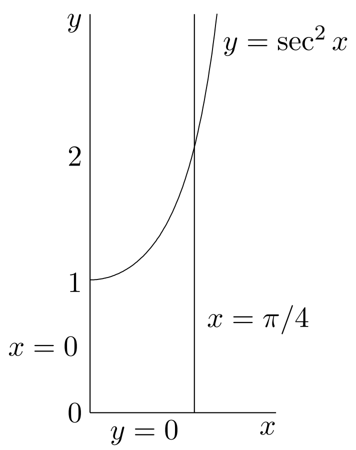
So at (x_1, y_1) the slope is

$$m = \frac{\sqrt{1-(1/\sqrt{2})^2}}{1/\sqrt{2}} = 1.$$

Using the formula $y - y_1 = m(x - x_1)$, this gives an equation of the tangent line:

$$y - \pi/4 = x + (\ln 2)/2 \quad \text{or} \quad y = x + (\pi + 2\ln 2)/4.$$

7.



The area A of the region is

$$A = \int_0^{\pi/4} \sec^2 x\,dx = \tan x|_0^{\pi/4} = 1.$$

The coordinates $(\bar{x}, \bar{y})$ of the centroid are given by

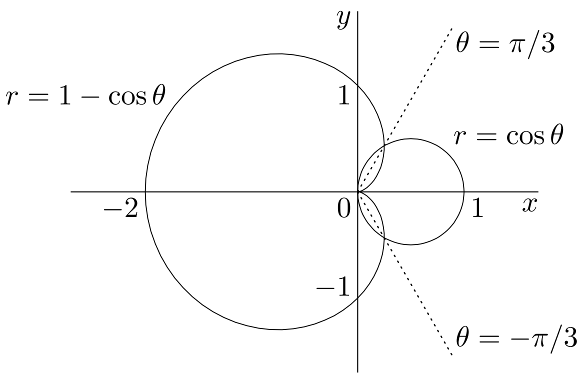
$$\bar{x} = \frac{1}{A}\int_0^{\pi/4} x\sec^2 x\,dx \quad \text{and} \quad \bar{y} = \frac{1}{2A}\int_0^{\pi/4} \sec^4 x\,dx.$$

To find $\bar{x}$, integrate by parts with $u = x$ and $dv = \sec^2 x\,dx$. So $du = dx$ and $v = \tan x$. This gives

$$\begin{aligned}\bar{x} &= x\tan x|_0^{\pi/4} - \int_0^{\pi/4} \tan x\,dx = \frac{\pi}{4} + \ln\cos x|_0^{\pi/4} = \frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right) \\ &= (\pi - 2\ln 2)/4.\end{aligned}$$

$$\bar{y} = \frac{1}{2}\int_0^{\pi/4} (1+\tan^2 x)\sec^2 x\,dx = \frac{1}{2}\left(\tan x + \frac{1}{3}\tan^3 x\right)\Bigg|_0^{\pi/4} = \frac{2}{3}.$$

8. The curves are sketched below.



The curves intersect when $1 - \cos\theta = \cos\theta \Longrightarrow \cos\theta = 1/2 \Longrightarrow \theta = \pm\pi/3$. The sketch shows that the area is

$$\begin{aligned}&2\left[\int_{\pi/3}^{\pi} \frac{1}{2}(1-\cos\theta)^2\,d\theta - \int_{\pi/3}^{\pi/2} \frac{1}{2}\cos^2\theta\,d\theta\right] \\ &= \int_{\pi/3}^{\pi} (1 - 2\cos\theta)\,d\theta + \int_{\pi/2}^{\pi} \cos^2\theta\,d\theta \\ &= (\theta - 2\sin\theta)|_{\pi/3}^{\pi} + \frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos 2\theta)\,d\theta \\ &= \frac{2\pi}{3} + \sqrt{3} + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_{\pi/2}^{\pi} = \frac{2\pi}{3} + \sqrt{3} + \frac{\pi}{4} = \frac{11\pi}{12} + \sqrt{3}.\end{aligned}$$